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Scaling Laws for all Liapunov Exponents: Models and Measurements

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We introduce simple diamond models of random symplectic matrices in order to study the scaling laws of all Liapunov exponents. These universal properties appear in physical problems that are modeled by transfer matrices: dynamical systems, random potentials, random fields, etc. Numerical experiments for the general case are in agreement with the results derived from the models.

KEY WORDS: Random matrices; Liapunov exponents; symplectic maps; diamond model.

1. INTRODUCTION

We consider the following symplectic map:

$$\begin{aligned} q_{n+1} &= q_n + p_n \\ p_{n+1} &= p_n + \varepsilon f(q_{n+1}) \end{aligned} \tag{1}$$

where q_n, $p_n \in \mathbf{R}^k$, and f is a smooth function. It can be considered as the Poincaré map of a Hamiltonian system. The corresponding tangent map is a $2k \times 2k$ symplectic matrix–valued map of the form

$$\mathbb{A}_n(\varepsilon) = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \varepsilon A_n & \mathbf{1} + \varepsilon A_n \end{pmatrix} \tag{2}$$

where $\mathbf{1}$ is the $k \times k$ identity matrix and A_n is the Jacobian of f at $\xi_n = (q_n, p_n)$. For these purposes it is therefore natural to consider symmetric matrices A_n whose nonzero elements are only the a_{ij} with $|i-j| \leqslant 1$,

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and $a_{k1} = a_{1k}$. The ordered product of those matrices naturally appear in this way. Now, we replace the product of the matrices along the orbit by a product of random matrices of the form (2), this can be considered as an approximation of the dynamics that turns out to present the correct behavior in certain regimes.[1-3]

The scaling behavior of the maximum Liapunov exponent as a function of ε for the infinite product of random matrices of the form (2) has been investigated numerically[4,5] as well as analytically.[6] These authors obtain $\lambda_{\max} \propto \varepsilon^{\beta}$ with either $\beta = 1/2$ or $\beta = 2/3$, depending on whether the common mean value of the elements of A_n is positive or zero.

Here we present a simple model that allows us to understand the behavior of all the Liapunov exponents. The result is also of interest for the localization properties in solid state physics. The discrete Schrödinger equation with random potentials for a strip geometry gives rise to products of matrices of the same type. Numerical experiments show that our models reproduce all known possible scaling behaviours of Liapunov exponents.

Considering the random process as an ergodic stationary process, we recall that Liapunov exponents can be defined as the logarithm of the eigenvalues of the matrix,

$$\Lambda = \lim_{m \to \infty} [\mathbb{M}_m(\varepsilon)^* \, \mathbb{M}_m(\varepsilon)]^{1/2m} \tag{3}$$

for a set of measure one of the parameters in the random space. Here

$$\mathbb{M}_m(\varepsilon) = \mathbb{A}_m(\varepsilon) \, \mathbb{A}_{m-1}(\varepsilon) \cdots \mathbb{A}_1(\varepsilon) \tag{4}$$

It is well known that these exponents are related to the rate of exponential growth of the volume forms in phase space.

Our numerical computations show that all the Liapunov exponents but the first generically scale with an exponent $\beta = 2/3$. In the next section we introduce a model that is able to reproduce these scaling laws. It is a "zeroth-order" model in the sense that it eliminates the possible couplings between eigenspaces, allowing us to perform a block reduction and to treat the problem as independent intermittent-type models.[7]

The model gives a hint for the construction of peculiar interactions among eigenspaces that modify the quoted $\beta = 2/3$ law. In fact, it is possible to devise models irreducible to a block matrix form, which give $\beta = 1/2$ law for some smaller Liapunov exponents (see Fig. 2).

2. THE DIAMOND MODEL

Let us begin with the simplest case of the matrices of the form (2) in the case $k = 2$, i.e., 4×4 matrices. We call a *diamond* a symmetric matrix with identical elements on the diagonal. We take the A_n to be a diamond random sequence, i.e.,

$$A_n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix} \tag{5}$$

where a_n, b_n are independent, identically distributed random variables. Note that, since the A_n in (2) must be symmetric, the only restriction of diamonds with respect to the general case is the equality of the two diagonal terms. It is clear from the definitions that no change is introduced in this study if we perform n-independent unitary transformations:

$$\mathbb{A}_n(\varepsilon) \to \hat{\mathbb{A}}_n(\varepsilon) = \mathbb{U}^{-1}\mathbb{A}_n(\varepsilon)\,\mathbb{U} \tag{6}$$

On the other hand, if V is a 2×2 unitary matrix, then $\mathbb{U} = V \otimes \mathbf{1}$ is a 4×4 unitary matrix, and the corresponding action on the matrices of the form (2) reduces to the change

$$A_n \to \tilde{A}_n = V^{-1} A_n V \tag{7}$$

We can therefore perform a first transformation taking

$$A_n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix}$$

into

$$\tilde{A}_n = \begin{pmatrix} s_n & 0 \\ 0 & r_n \end{pmatrix} \tag{8}$$

where $s_n = a_n + b_n$ and $r_n = a_n - b_n$ are again independent random variables with the mean value of s_n zero only if the two variables are of zero mean, but with the mean value of r_n equal to zero.

A second unitary transformation takes $\tilde{\mathbb{A}}_n(\varepsilon)$ into the following block matrix:

$$\mathbb{A}_n = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \varepsilon s_n & 1+\varepsilon s_n & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \varepsilon r_n & 1+\varepsilon r_n \end{pmatrix} \tag{9}$$

As we can see, each diagonal block matrix is again of the form (2). Now it is clear what happens in this case. Following ref. 6, if the mean values of a_n and b_n are zero, so is the mean value of s_n and, since we have already said that the mean value of r_n is always zero, we get a common law for the two (positive) Liapunov exponents of the form $\lambda \propto \varepsilon^{2/3}$. If, instead, the mean value of the a_n and b_n is nonzero, we get a maximum Liapunov with a scaling as $\varepsilon^{1/2}$, but a second Liapunov scaling as $\varepsilon^{2/3}$. It is not difficult to obtain $2k$-dimensional models that reduce after a convenient transformation to the block form (9) of the matrices.

To be more explicit, we define a diamond for $k>2$ (k even) as the tensor product of 2×2 diamonds, since it is then clear that we can again diagonalize the A_n by means of an n-independent unitary transformation. Furthermore, the well-known formula of the eigenvalues of a tensor product shows that their means follow the same simple rule as in the case of 2×2 diamonds.

Once one has reduced the $k\times k$ matrix A_n to a diagonal form with random elements by some $k\times k$ unitary transformation, it can be seen straightforwardly that the $2k\times 2k$ matrix $\mathbb{U}$ with elements

$$u_{ij}=0$$

but

$$u_{i,2i-1}=1 \qquad \text{if} \quad i\leqslant k$$

$$u_{i,2(i-k)}=1 \qquad \text{if} \quad i>k$$

takes it into a matrix with 2×2 blocks as in (9). Therefore, composing random variables with zero and nonzero means, it is easy to have examples of models with the j first Liapunov exponents having a scaling law $\lambda \propto \varepsilon^{1/2}$ ($0\leqslant j\leqslant k$) and the remaining $k-j$ with the scaling law $\lambda \propto \varepsilon^{2/3}$. The fact that for small ε the Liapunov exponents with these scaling laws arrange in this order comes from the law itself.

3. REMARKS ON THE GENERAL CASE

We come back for a moment to the (4×4)-dimensional case. It is known that the area of a generic parallelogram in $\mathbf{R}^4$ asymptotically grows as the exponential of $m(\lambda_1+\lambda_2)$ as the map (2) iterates m times, the λ_i being the two positive Liapunov exponents. Therefore, up to a renormalization, the second Liapunov exponent is given by the asymptotic behavior of the projection of the iterate of one initial vector in a direction normal to the iterate of another vector. In higher dimension, taking a suitable number of independent vectors, the same type of construction gives each of the remaining Liapunov exponents.

Taking a collection of n linearly independent vectors, our model suggests that the process corresponding to the projection of a vector in the normal direction to the hyperplane containing the preceding vectors is generically a process of zero mean. Numerical computations confirm this suggestion, allowing to verify quantitatively that all projections in fact fluctuate with zero mean.

The most general situation corresponds to the case where there is some coupling between eigenspaces, which amounts to having nonzero off-diagonal blocks in (9). Numerical simulations show that in this case, at least concerning scaling laws, this fact introduces only small fluctuations and that therefore our "zeroth-order" model is a crude but significant approximation of the generic case: the asymptotic behavior of the complicated process corresponding to the product of symplectic random matrices should be as like the diamond independence, with zero-mean intermittent processes in all but one block.

4. NUMERICAL EXPERIMENTS

We have computed the Liapunov exponents up to the case of 8×8 matrices, using a well-established procedure,[8] with "time" relaxation of

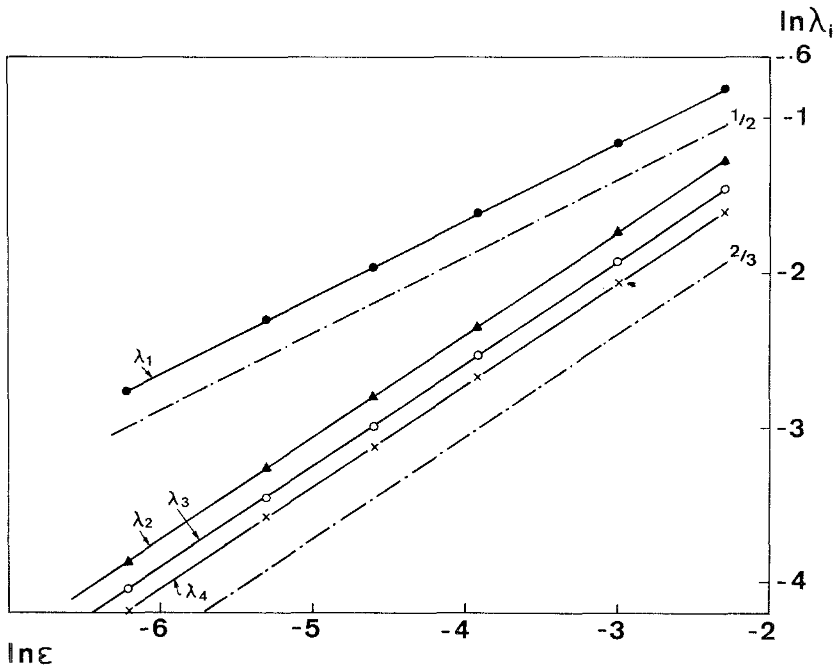

Fig. 1. Scaling laws for the Liapunov exponents of products of random symplectic 8×8 matrices in the general case as a function of the perturbation parameter ε on a log–log scale. The slopes 1/2 and 2/3 are reported for comparison. For graphical purposes the data have been rescaled: (▲) $a\lambda_2$, (○) $b\lambda_3$, (×) $c\lambda_4$, with $a = e^{1.5}$, $b = e^{1.8}$, $c = e^{2.4}$.

order of 10^7 iterations and corresponding stabilization tests. The elements of the A_n matrix were chosen with a uniform probability distribution within a length-one interval centered around the mean value. However, it was verified that, in general, the choice of probability distribution does not affect the scaling laws apart from the first moment, which modifies only the scaling law of the maximum Liapunov exponent, as said before.

However, particular choices of the correlations among the elements of the A_n matrices give rise to anomalous scaling laws for the Liapunov exponent spectrum. We reproduce only two of these results, since there are no significant differences for all the remaining experiments.

In Fig. 1, we see that the scaling laws in the 8×8 general case are in a very good agreement with the prediction of the model in the case that all but one of the random variables $s_n(i)$, $i = 1, 2, \ldots, k$, are of the "difference type," i.e., of zero mean.

Another interesting case is that where the diagonal elements of the A_n are taken to be the sum of the row elements. This leads after unitary trans-

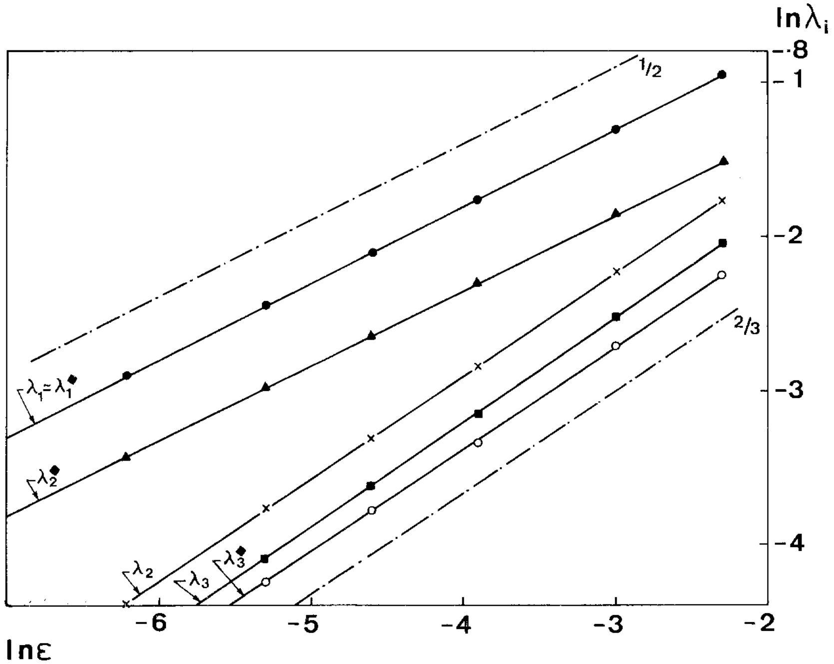

Fig. 2. Scaling laws for the Liapunov exponents of products of random symplectic 6×6 matrices: the general case λ_1, λ_2, λ_3 and generalized diamond $\lambda_1^\blacklozenge$, $\lambda_2^\blacklozenge$, $\lambda_3^\blacklozenge$. The maximum exponents differ only slightly, but $\lambda_2^\blacklozenge$ shows a 1/2 scaling law instead of 2/3 for λ_2. Again the data have been rescaled for graphical purposes: (×) $e^{1.2}\lambda_2$, (■) $e^{1.8}\lambda_3$, (○) $e^{1.2}\lambda_3^\blacklozenge$. The slopes 1/2 and 2/3 are reported for comparison.

formation of the type introduced in Section 2 to a generalized diamond model (diamond flush) in the presence of n-independent coupling among eigenspaces. Whereas in the general case of matrices of type (2), as expected from the previous analysis, only $\lambda_{\max} = \lambda_1$ has a scaling law as $\varepsilon^{1/2}$, in the case of the generalized diamond, by choosing appropriately the mean of independent elements, we are able to produce two Liapunov exponents with the same scaling law $\varepsilon^{1/2}$, as we show in Fig. 2 for the case of 6×6 matrices.

5. CONCLUSION

Diamond models produce all known scaling laws for the Liapunov exponents of products of random symplectic matrices. They even suggest how to introduce correlations among the elements of the matrices such as to modify the scaling laws (as done for the diamond flush). These results can be easily generalized to matrices of a form different from (2), for instance, transfer matrices of a Schrödinger operator on a strip lattice at the band edge.

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